Scattering by a single particle

A parallel beam of light with a flux density $F$ along the incident direction $(\theta_i, \phi_i)$ shines on a particle. The scattered light is measured within a small cone of solid angle $d\Omega$, along a direction which makes an angle $\gamma$ with the incident beam. The angle $\gamma$ is known as the \textbf{scattering angle}. The \textbf{intensity} of the scattered light (i.e. the scattered flux per unit solid angle) along the direction $(\theta, \phi)$ is,

$$I(\theta, \phi) = \frac{d\Phi}{d\Omega} \text{ [W sr}^{-1}]$$ \hspace{1cm} (1)

For simplicity of presentation, we will assume for the time being that the particle is a homogeneous sphere, and to simplify the scattering geometry, the $z$-axis is directed along the direction of the incident flux, with the origin at the centre of the particle. The scattering angle $\gamma$ is then equal to the angle $\theta$ of the spherical coordinate system. By azimuthal symmetry, the scattered intensity depends only on the scattering angle $\gamma$.

The parameter that relates the scattered intensity to the incident flux density is the \textbf{angular scattering cross-section}:

$$\sigma(\theta) = \frac{I(\theta)}{F} \text{ [m}^2 \text{ sr}^{-1}]$$ \hspace{1cm} (2)

The \textbf{total scattering cross-section} is the total flux scattered into all directions per unit incident flux density:

$$\sigma_s = \int_0^\pi \int_0^{2\pi} \sigma(\theta) \sin \theta \, d\theta \, d\phi = 2\pi \int_0^\pi \sigma(\theta) \sin \theta \, d\theta \text{ [m}^2]$$ \hspace{1cm} (3)

So the total scattered flux is,

$$q_s = F \sigma_s \text{ [W]} \hspace{1cm} (4)$$
The **scattering efficiency** is the ratio of the scattering cross-section $\sigma_s$ to the geometric cross-section $A_g$ of the particle:

$$Q_s = \frac{\sigma_s}{A_g} \quad \text{[dimensionless]} \quad (5)$$

**Absorption**

If the refractive index of the scattering particle has non-zero imaginary part, then absorption occurs. The total power absorbed per unit incident flux density is defined as the **absorption coefficient** $\sigma_a$.

The total absorbed flux is,

$$q_a = F \sigma_a \quad \text{[W]} \quad (6)$$

and the **absorption efficiency** is

$$Q_a = \frac{\sigma_a}{A_g} \quad \text{[dimensionless]} \quad (7)$$

The **Extinction Cross-section** is the sum of the scattering and absorption cross-sections:

$$\sigma_e = \sigma_s + \sigma_a \quad (8)$$

while the **Extinction Efficiency** is the sum of the scattering and absorption efficiencies:

$$Q_e = Q_s + Q_a \quad (9)$$

**Single Scattering Albedo**

The single scattering albedo of the particle is defined as the ratio of the scattering cross-section to extinction cross-section:

$$\omega = \frac{\sigma_s}{\sigma_e} = \frac{\sigma_s}{\sigma_s + \sigma_a} \quad \text{(10)}$$

Its value ranges from 0 (totally absorbing particle, no scattering) to 1 (totally scattering particle, no absorption).

**Scattering Phase Function**

The scattering phase function is the ratio of the angular scattering cross-section to the total scattering cross-section, multiplied by a factor $4\pi$:

$$P(\theta) = 4\pi \frac{\sigma(\theta)}{\sigma_s} \quad (11)$$

Note that in the case of azimuthal symmetry considered here, the phase function is dependent only on the scattering angle $\theta$. The scattering phase function describes the directional distribution of the scattered flux. It is normalized such that

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} P(\theta) \sin \theta \, d\theta \, d\phi = 1 \quad (12)$$
Asymmetricity parameter of the scattering phase function

The forward scattering direction refers to scattering angle $0 < \theta < \pi/2$ while the backward scattering direction refers to the range of scattering angles $\pi/2 < \theta < \pi$. Typically, for a small particle (i.e. small relative to the wavelength $\lambda$), the phase function is symmetric in the forward and backward direction. For a larger particle, the phase function is peaked towards the forward scattering direction, i.e. the phase function has large values for small scattering angle. The parameter that characterize the asymmetricity of the phase function is the mean value of the cosine of the scattering angle, using the phase function as the weighting function.

$$g = \frac{\int^{-1}_{1} P(\theta) \cos \theta \, d(\cos \theta)}{\int^{-1}_{1} P(\theta) \, d(\cos \theta)}$$  

Scattering by a Suspension of Particles in Air

We now consider the case where light is scattered by a collection of particles suspended in a medium (such as aerosol particles suspended in air). The distance between the particles are assumed to be large compared to the wavelength of the incident light, so that the intensities scattered by the various particles can be added up without considering interference effect (Independent Incoherent Scattering).

Mono-Dispersion of Particles

We will first consider a special case where all particles are identical, with the same size and composed of the same material. Let the particle density be $N$ (no. of particles in a unit volume of air).

The intensity scattered from a small volume $V$ of the scattering medium can be expressed as

$$I(\theta) = FVN \sigma(\theta)$$  

where $F$ is the incident flux density and $\sigma(\theta)$ is the angular scattering cross-section of a single particle, $\theta$ is the scattering angle. The angular scattering coefficient of the scattering medium is the product of particle density and angular scattering cross-section:

$$\beta(\theta) = N \sigma(\theta) \ \text{[m}^{-1} \ \text{sr}]$$  

The scattering coefficient, absorption coefficient and extinction coefficient can be similarly defined:

**Scattering Coefficient:** $\beta_s = N \sigma_s \ \text{[m}^{-1}]$  

**Absorption Coefficient:** $\beta_a = N \sigma_a \ \text{[m}^{-1}]$  

**Extinction Coefficient:** $\beta_e = N \sigma_e \ \text{[m}^{-1}]$  

$$\beta_s + \beta_a = \beta_e$$
The singular scattering albedo is
\[ \omega = \frac{\beta_s}{\beta_e} \] (20)

and the scattering phase function is
\[ P(\theta) = 4\pi \frac{\beta(\theta)}{\beta_s} \] (21)

**Polydispersion of Particles**

Now we consider a more general case where the scattering particles has a size distribution \( n(r) \) such that \( n(r)dr \) is the number of particles with radius ranging from \( r \) to \( r + dr \) per unit volume of the scattering medium. The minimum and maximum radii of the particles are \( r_{\text{min}} \) and \( r_{\text{max}} \) respectively.

The angular scattering coefficient is now expressed as
\[ \beta(\theta) = \frac{r_{\text{max}}}{r_{\text{min}}} \int n(r)\sigma(\theta; r) \, dr \] (22)

Note that the angular scattering coefficient is a function of the scattering angle \( \theta \) and the particle radius \( r \).

The scattering, absorption and extinction coefficients are similarly defined as:

**Scattering Coefficient:** \( \beta_s = \int_{r_{\text{min}}}^{r_{\text{max}}} n(r)\sigma_s(r) \, dr \) (23)

**Absorption Coefficient:** \( \beta_a = \int_{r_{\text{min}}}^{r_{\text{max}}} n(r)\sigma_a(r) \, dr \) (24)

**Extinction Coefficient:** \( \beta_e = \int_{r_{\text{min}}}^{r_{\text{max}}} n(r)\sigma_e(r) \, dr \) (25)

\[ \beta_s + \beta_a = \beta_e \] (26)

The single scattering albedo and scattering phase function are expressed similarly in terms of the coefficients as in the monodispersion case.
Beer-Lambert Law

Consider a small volume of the scattering medium, of cross sectional area $A$ and thickness $\Delta z$, and hence with volume $V = A \Delta z$. A beam of light incident on to the scattering volume. The incident flux density is $F$ and the transmitted flux density emerging from the opposite side of the scattering volume is $F + \Delta F$. In general, $\Delta F$ is a negative quantity due to scattering and absorption. We assume the $z$-axis is pointing along the direction of the incident flux.

By generalizing expression (14), the angular intensity (flux per unit solid angle) scattered from this scattering volume $V$ of the scattering medium can be generalized to

$$I(\theta) = F V \beta(\theta)$$

(27)

where $\beta(\theta)$ is the angular scattering coefficient of the medium (Eq. 22).

The total scattered flux is obtained by integrating (27) for all directions,

$$\Phi_s = \int I(\theta) \sin\theta \, d\theta \, d\phi = F V \beta_s$$

(28)

where $\beta_s$ is the scattering coefficient (Eq. 23).

The total absorbed flux can be similarly expressed as

$$\Phi_a = F V \beta_a$$

(29)

The total flux removed from the incident beam due to scattering and absorption is

$$\Phi_e = \Phi_s + \Phi_a = F V (\beta_s + \beta_a) = F V \beta_e$$

(30)

Hence, the transmitted flux density is

$$\Delta F = -\frac{\Phi_e}{A} = -F \beta_e \Delta z$$

(31)

This is the differential form of the **Beer-Lambert Law**. If the extinction coefficient is constant, independent of the coordinate $z$, then the equation can be integrated to gives the exponential decay of the flux density along the direction of propagation,

$$F = F_0 \exp(-\beta_e z)$$

(32)

Reflectance, Absorptance and Transmittance of a Thin Layer of Air

Consider again the scattering volume of thickness $\Delta z$ in the previous section. Recall that the reflectance, absorptance and transmittance are defined respectively as the ratios of the reflected, absorbed and transmitted fluxes to the incident flux. The incident flux in this case is
\[ \Phi_i = FA \]  

Hence, from equations (29), the absorptance for the thin layer of air with thickness \( \Delta z \) is

\[ \alpha = \frac{\Phi_a}{\Phi_i} = \frac{FA \Delta z \beta_a}{FA} = \beta_a \Delta z \]  

(34)

The total flux scattered in the backward direction is

\[ \Phi_{sb} = \int_0^{2\pi/2} \int_0^{2\pi} I(\theta) \sin \theta \, d\phi = FA \Delta z \int_0^{2\pi/2} \beta(\theta) \sin \theta \, d\phi = FA \beta_{sb} \Delta z \]  

(35)

Here, we have defined the **backscattering coefficient** \( \beta_{sb} \) of the scattering medium to be

\[ \beta_{sb} = \int_0^{2\pi/2} \beta(\theta) \sin \theta \, d\phi \]  

(36)

Hence, the reflectance of the thin layer of air with thickness \( \Delta z \) is

\[ \rho = \frac{\Phi_{sb}}{\Phi_i} = \beta_{sb} \Delta z \]  

(37)

The transmittance has two components: the direct and diffused components. The direct transmittance is due to the removal of flux from the incident beam by absorption and scattering into all (forward and backward) directions, while the diffused transmittance is due to the scattering of the incident beam into all the forward directions.

The total flux scattered into the forward direction is

\[ \Phi_{sf} = \int_0^{2\pi/2} \int_0^{2\pi} I(\theta) \sin \theta \, d\phi = FA \Delta z \int_0^{2\pi/2} \beta(\theta) \sin \theta \, d\phi = FA \beta_{sf} \Delta z \]  

(38)

where the **forward scattering coefficient** \( \beta_{sf} \) is defined as

\[ \beta_{sf} = \int_0^{2\pi/2} \beta(\theta) \sin \theta \, d\phi \]  

(39)

The diffused transmittance due to the forward scattered flux is

\[ \tau_{\text{diffused}} = \frac{\Phi_{sf}}{\Phi_i} = \beta_{sf} \Delta z \]  

(40)

while the direct transmittance is

\[ \tau_{\text{direct}} = \frac{\Phi_i - (\Phi_a + \Phi_s)}{\Phi_i} = 1 - (\beta_a + \beta_s) \Delta z = 1 - \beta_e \Delta z \]  

(41)

Hence, the total transmittance is

\[ \tau = \tau_{\text{direct}} + \tau_{\text{diffused}} = 1 - (\beta_e - \beta_{sf}) \Delta z \]  

(42)

By conservation of energy, the transmittance \( \tau \), reflectance \( \rho \) and absorptance \( \alpha \) must sum up to one,

\[ \tau + \rho + \alpha = 1 \]  

(43)

Hence, we have

\[ 1 - (\beta_e - \beta_{sf}) \Delta z + \beta_{sb} \Delta z + \beta_a \Delta z = 1 \]  

(44)
Which leads to the relation
\[ \beta_e = \beta_{sf} + \beta_{sb} + \beta_a = \beta_s + \beta_a \]  
which is identical to Eqn. (26).

Note that we have assumed that the thickness \( \Delta z \) is very thin such that the flux scattered through the edge of the scattering volume is negligible compared to those scattered through the front and back surface.

### General Scattering Geometry

Let \( \mathbf{s}_i \) be a unit vector pointing along the direction \((\theta_i, \phi_i)\) of the incident flux and let \( \mathbf{s}_s \) be a unit vector pointing along the direction \((\theta_s, \phi_s)\) of the scattered flux. The incident radiance along \( \mathbf{s}_i \) is \( L(\theta_i, \phi_i) \). We consider the scattering events occurring in a small volume \( V \) of the scattering medium. The incident flux density within the solid angle \( d\Omega_i \) is
\[ dF_i = L(\theta_i, \phi_i) d\Omega_i \]  
and from eqn (27), the scattered intensity due to this incident flux density is
\[ dI(\theta_s, \phi_s) = L(\theta_i, \phi_i) V \beta(\theta_s, \phi_s; \theta_i, \phi_i) d\Omega_i \]  
Note that, in general, the scattered intensity depends on the directions of the incident and the scattered fluxes. This dependence is explicitly included in the arguments of the angular scattering coefficient.

Suppose that the incident radiance comes from all direction, then the scattered intensity per unit volume of scattering medium is obtained by integrating \( dI(\theta_s, \phi_s) \) over all directions of the incident radiance:
\[ \frac{I(\theta_s, \phi_s)}{V} = \int L(\theta_i, \phi_i) \beta(\theta_s, \phi_s; \theta_i, \phi_i) d\Omega_i \]  
Using the relations between the angular scattering coefficient, scattering coefficient and scattering phase function (eqn. 21), this expression can be rewritten as:
\[
\frac{I(\theta_s, \phi_s)}{V} = \frac{\beta_s}{4\pi} \int_0^\infty \int_0^\infty L(\theta_i, \phi_i) P(\theta_s, \phi_s; \theta_i, \phi_i) \sin \theta_i \, d\theta_i \, d\phi_i
\]  

(49)

If the scattering medium is homogeneous, then the scattering phase function depends only on the scattering angle \(\gamma\),

\[P(\theta_s, \phi_s; \theta_i, \phi_i) = P(\gamma)\]  

(50)

The scattering angle \(\gamma\) can be computed by noting that \(\cos \gamma\) is the dot-product of the incident vector \(s_i\) and the scattering vector \(s_r\),

\[\cos \gamma = s_i \cdot s_r\]  

(51)

and it can be shown that,

\[
\cos \gamma = \cos \theta_i \cos \theta_r + \sqrt{(1 - \cos^2 \theta_i)} \sqrt{(1 - \cos^2 \theta_r)} \cos(\phi_i - \phi_r)
\]  

(52)

NOTES:

In this chapter, all equations have been written in terms of the radiant quantities, without explicitly referring to the dependence on wavelength. It should be apparent that the equations are still valid if the radiant quantities are replaced by the respective spectral radiant quantities.