Consider a small volume of atmosphere located at the position \( r(x, y, z) \), with an infinitesimal thickness \( ds \) and cross-sectional area \( A \). The normal to the area points in the direction \( s(\theta, \phi) \). Consider a ray of light with a spectral radiance \( L_{\lambda}(r, s) \) incident into the volume along the direction \( s \). The radiance after emerging from the volume of atmosphere along the same direction \( s \) is \( L_{\lambda}(r, s) + dL_{\lambda} \).

There are three components that contribute to the change in radiance \( dL_{\lambda} \).

**Extinction of the direct beam**

The extinction of the direct beam due to scattering and absorption within the scattering volume results in a decrease of the radiance according to the Beer-Lambert Law:

\[
dL_{\lambda, \text{direct}} = -L_{\lambda}(r, s) \beta_{e}(r) \: ds
\]  

(1)

**Scattering**

The flux along the direction \( s \) increases due to scattering of flux coming from other directions \( s'(0', \phi') \) to the direction \( s \). The increase in flux due to scattering is

\[
dL_{\lambda, \text{scattered}} = \frac{\beta_{s}(r)}{4\pi} \int_{\text{all directions}} L_{\lambda}(r, s') P(s, s', r, \lambda) \: d\Omega'
\]  

(2)

**Thermal emission**

The volume of air at a temperature \( T \) emits blackbody radiation according to the Planck Law, resulting to an increase in the radiance along the direction \( s \):

\[
dL_{\lambda, \text{thermal}} = \varepsilon_{\lambda}(r)L_{\lambda}^{(B)}(T(r))
\]  

(3)

For a thin layer of the air at thermal equilibrium, the emissivity is equal to the absorptance which is related to the absorption coefficient and the thickness of the layer of air,
Hence, the increase in radiance due to thermal emission is
\[
dL_{\lambda, \text{thermal}} = \beta_a(\mathbf{r}, \lambda) L_{\lambda}(^B)(T(\mathbf{r})) \, ds
\]

**Radiative Transfer Equation (RTE)**

Putting the three components together, the change in radiance along the direction \(s\) is
\[
dL_{\lambda} = -\beta_e(\mathbf{r}, \lambda)L_{\lambda}(\mathbf{r}, s)ds + \frac{\beta_s(\mathbf{r}, \lambda)}{4\pi} \int P(s, s'; \mathbf{r}, \lambda)L_{\lambda}(\mathbf{r}, s') \, d\Omega' \, ds + \beta_a(\mathbf{r}, \lambda)L_{\lambda}(^B)(T(\mathbf{r})) \, ds
\]

**Notes:**

1. If the material within the scattering volume is homogeneous, then the scattering phase function is dependent only on the scattering angle \(\gamma\), given by
\[
\cos(\gamma) = \mathbf{s}' \cdot \mathbf{s}
\]

2. The absorption, scattering and extinction coefficients are related by
\[
\beta_e = \beta_s + \beta_a
\]

Recall that the single scattering albedo is the ratio of the scattering to the extinction coefficients:
\[
\omega = \frac{\beta_s}{\beta_e}
\]

Hence, the scattering and absorption coefficients can be expressed in terms of the extinction coefficient and the single scattering albedo,
\[
\beta_s = \omega \beta_e; \quad \beta_a = (1 - \omega) \beta_e
\]

Dividing both sides of Eqn. (6) by \(\beta_e(\mathbf{r}, \lambda) ds\) and rearranging, we get the radiative transfer equation (RTE):
\[
- \frac{1}{\beta_e(\mathbf{r}, \lambda)} \frac{dL_{\lambda}(\mathbf{r}, s)}{ds} = L_{\lambda}(\mathbf{r}, s) - J_{\lambda}(\mathbf{r}, s)
\]

The RTE has a **source function** \(J_{\lambda}(\mathbf{r}, s)\) that is the sum of two terms: a **scattering** source function \(J_{\lambda,s}(\mathbf{r}, s)\) and an **emission** source function \(J_{\lambda,e}(\mathbf{r}, s)\),
\[
J_{\lambda}(\mathbf{r}, s) = J_{\lambda,s}(\mathbf{r}, s) + J_{\lambda,e}(\mathbf{r}, s)
\]
The thermal emission source function is

\[ J_{\lambda,E}(r,s) = [1 - \omega(r,\lambda)]L_{\lambda}^B(T(r)) \]  

\[ (14) \]

Radiative Transfer Equation in Vector Notations

The RTE can be written in a compact form as:

\[ -\frac{1}{\beta_e(r,\lambda)}(s \cdot \nabla)L_{\lambda}(r,s) = L_{\lambda}(r,s) - J_{\lambda}(r,s) \]  

\[ (15) \]

Radiative Transfer Equation in the Cartesian Coordinates System

In the Cartesian coordinate system, the RTE has the form:

\[ -\frac{1}{\beta_e(x,y,z,\lambda)}\left[ \cos \theta_x \frac{\partial}{\partial x} + \cos \theta_y \frac{\partial}{\partial y} + \cos \theta_z \frac{\partial}{\partial z} \right]L_{\lambda}(x,y,z;s) = L_{\lambda}(x,y,z;s) - J_{\lambda}(x,y,z;s) \]  

\[ (16) \]

where \( \theta_x, \theta_y, \) and \( \theta_z \) are the angles between the directional vector \( s \) and the \( x-, y- \) and \( z-\) axes respectively.
RTE for a Plane-Parallel Atmosphere

A thin horizontal layer of atmosphere above the ground,
Height $z > 0$,
Optical Thickness $0 < \tau < \tau_0$

When viewed from the top of the atmosphere, it is more convenient to deal with the optical thickness, instead of the physical thickness of the atmosphere. The optical thickness at height $z$ as seen from the top of atmosphere is defined as,

$$\tau(z, \lambda) = \int_0^\infty \beta_e(z', \lambda) dz'$$

When defined in this way, the optical thickness is a decreasing function of the height above the ground. At the top of the atmosphere ($z = \infty$), the optical thickness is zero. At the ground level ($z = 0$), the total optical thickness of the atmosphere is

$$\tau_0(\lambda) = \int_0^\infty \beta_e(z', \lambda) dz'$$
It is also more convenient to work with the cosine of the zenith angle,

\[ u = \cos(\theta) \]  

(20)

Note that \( u \) is positive \((0 < u < 1)\) for upward direction \((0 < \theta < \pi/2)\) and negative \((-1 < u < 0)\) for downward direction \((\pi/2 < \theta < \pi)\).

Using the new coordinates \( \tau \) and \( u \), the RTE for the plane parallel atmosphere can be expressed in the form

\[ u \frac{d}{d\tau} L_\lambda (\tau; u, \phi) = L_\lambda (\tau; u, \phi) - J_\lambda (\tau; u, \phi) \]  

(21)

The source functions are:

\[ J_{\lambda, S} (\tau, u, \phi) = \frac{\omega(\tau, \lambda)}{4\pi} \int_{0}^{1} \int_{-1}^{1} P(u, \phi, u', \phi', \tau, \lambda) L_\lambda (\tau; u', \phi') \, du' \, d\phi' \]  

(22)

\[ J_{\lambda, E} (\tau, u, \phi) = [1 - \omega(\tau, \lambda)] L_\lambda^B (T(\tau)) \]  

(23)